ON THE ENDOMORPHISM ALGEBRAS OF MODULAR GELFAND-GRAEV REPRESENTATIONS

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This paper is dedicated to Toshiaki Shoji, on his sixtieth birthday.

Abstract. We study the endomorphism algebras of a modular Gelfand-Graev representation of a finite reductive group by investigating modular properties of homomorphisms constructed by Curtis and Curtis-Shoji.

Let $\mathbf{G}$ be a connected reductive group defined over an algebraic closure $\mathbb{F}$ of the field of p-elements $\mathbb{F}_p$ and suppose that it is endowed with a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ relative to an $\mathbb{F}_q$-structure. Since the work of Lusztig, it has been natural to ask to what extent the theory of the representations of $\mathbf{G}^F$ depends on q. For example, it was shown by Lusztig that the unipotent characters of $\mathbf{G}^F$ are parametrized by a set which is independent of q (the set depends solely on the Weyl group of $\mathbf{G}$ and on the action of F on this Weyl group).

On the side of ℓ-modular representations (where ℓ is a prime different from p), the work of Fong and Srinivasan on the general linear and unitary groups [FS1] and on the classical groups [FS2], then that of Broué, Malle and Michel (introducing the notion of *generic groups* [BMM]) and of Cabanes and Enguehard [CE1] give evidence of analogous results. For instance, in most cases, the unipotent ℓ-blocks of $\mathbf{G}^F$ only depend on the order of q modulo ℓ, and not on the value of q itself [CE2, Chapter 22].

Let $(K, \mathcal{O}, k)$ denote an ℓ-modular system, sufficiently large. In this article we will study the endomorphism algebra $\mathcal{H}_{(d)}^{\mathbf{G}}$ of a modular Gelfand-Graev representation $\Gamma_{(d)}^{\mathbf{G}}$ of $\mathbf{G}^{F^d}$ (this is a projective $\mathcal{O}\mathbf{G}^{F^d}$-module). We will study also the corresponding unipotent parts $b_{(d)}^{\mathbf{G}}\mathcal{H}_{(d)}^{\mathbf{G}}$ and $b_{(d)}^{\mathbf{G}}\Gamma_{(d)}^{\mathbf{G}}$ (here, $b_{(d)}^{\mathbf{G}}$ denotes the sum of the unipotent blocks of $\mathcal{O}\mathbf{G}^{F^d}$). We make the following conjecture, which is related to the question mentioned above:

Conjecture 1. *If ℓ does not divide $[\mathbf{G}^{F^d} : \mathbf{G}^F]$, then the $\mathcal{O}$-algebras $b_{(d)}^{\mathbf{G}}\mathcal{H}_{(d)}^{\mathbf{G}}$ and $b_{(1)}^{\mathbf{G}}\mathcal{H}_{(1)}^{\mathbf{G}}$ are isomorphic.*

Date: April 25, 2008.
1991 *Mathematics Subject Classification.* According to the 2000 classification: Primary 20C20; Secondary 20G40.

The $\mathcal{O}\mathbf{G}^{F^d}$-module $b_{(d)}^{\mathbf{G}}\Gamma_{(d)}^{\mathbf{G}}$ is projective and indecomposable : it is the projective cover of the modular *Steinberg module*. Conjecture 1, if proven, would show that the endomorphism algebra of this module does not depend *too much* on q.

In this article, we approach Conjecture 1 by the study of a morphism ${}_K\mathrm{Cur}_{\mathbf{L},(d)}^{\mathbf{G}} : K\mathcal{H}_{(d)}^{\mathbf{G}} \to K\mathcal{H}_{(d)}^{\mathbf{L}}$ (where $\mathbf{L}$ is an F^d-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}$). When $\mathbf{T}$ is a maximal F^d-stable torus of $\mathbf{G}$, this morphism was constructed by Curtis [C, Theorem 4.2] and it is defined over $\mathcal{O}$ (i.e. there exists a morphism of algebras $\mathrm{Cur}_{\mathbf{T},(d)}^{\mathbf{G}} : \mathcal{H}_{(d)}^{\mathbf{G}} \to \mathcal{H}_{(d)}^{\mathbf{T}} = \mathcal{O}\mathbf{T}^{F^d}$ such that ${}_K\mathrm{Cur}_{\mathbf{T},(d)}^{\mathbf{G}}$ is obtained from $\mathrm{Cur}_{\mathbf{T},(d)}^{\mathbf{G}}$ by extension of scalars). We will also consider a product of Curtis homomorphisms

$$\mathrm{Cur}_{(d)}^{\mathbf{G}} : \mathcal{H}_{(d)}^{\mathbf{G}} \longrightarrow \prod_{\mathbf{T}\in\boldsymbol{\mathcal{T}}^{F^d}} \mathcal{O}\mathbf{T}^{F^d},$$

where $\boldsymbol{\mathcal{T}}$ is the variety of maximal tori of $\mathbf{G}$. Finally, we will study a morphism of K-algebras ${}_K\Delta^{\mathbf{G}} : K\mathcal{H}_{(d)}^{\mathbf{G}} \to K\mathcal{H}_{(1)}^{\mathbf{G}}$ defined by Curtis and Shoji [CS, Theorem 1]. With this notation, we can state Conjecture 1 more precisely :

Conjecture 2. *With the notation above, we have:*
(a) ${}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}$ *is defined over* $\mathcal{O}$.
(b) ${}_K\Delta^{\mathbf{G}}$ *is defined over* $\mathcal{O}$.
(c) *If* ℓ *does not divide* $[\mathbf{G}^{F^d} : \mathbf{G}^F]$, *then* ${}_K\Delta^{\mathbf{G}}$ *induces an isomorphism* $b_{(d)}^{\mathbf{G}}\mathcal{H}_{(d)}^{\mathbf{G}} \simeq b_{(1)}^{\mathbf{G}}\mathcal{H}_{(1)}^{\mathbf{G}}$.

The main results of this article are obtained under the hypothesis that ℓ does not divide the order of the Weyl group W of $\mathbf{G}$.

Theorem. *If* ℓ *does not divide* $|W|$, *then Conjecture 2 holds.*

Statement (a) is proved in Corollary 3.12 ; statement (b) in Theorem 4.4 ; statement (c) is shown in Theorem 4.9. In order to obtain our theorem, we proved two more precise results which do not necessarily hold when ℓ does divide $|W|$.

Theorem 3.7. *If* ℓ *does not divide* $|W|$, *then*

$$\mathrm{Im}(\mathrm{Cur}_{(d)}^{\mathbf{G}}) = \mathrm{Im}({}_K\mathrm{Cur}_{(d)}^{\mathbf{G}}) \cap \prod_{\mathbf{T}\in\boldsymbol{\mathcal{T}}^{F^d}} \mathcal{O}\mathbf{T}^{F^d}.$$

Theorem 3.13. *If* ℓ *does not divide* $|W|$, *then*

$$b_{(d)}^{\mathbf{G}}\mathcal{H}_{(d)}^{\mathbf{G}} \simeq (\mathcal{O}S)^{N_{\mathbf{G}^{F^d}}(S)},$$

where S *is a Sylow* ℓ*-subgroup of* $\mathbf{G}^{F^d}$.

REMARK- With the above notation, if ℓ does not divide $|W|$, then S is abelian, and hence a consequence of the above result is that if ℓ does not divide $|W|$, then the

isomorphism type of the $\mathcal{O}$-algebra $b^{\mathbf{G}}_{(d)}\mathcal{H}^{\mathbf{G}}_{(d)}$ depends only on the fusion of ℓ-elements in $\mathbf{G}^{F^d}$.

This article is organized as follows. In the first section, we recall the definitions of the Gelfand-Graev representations as well as some of the principal properties of their endomorphism algebras (commutativity for example). In the second section, we construct the generalisation of the Curtis homomorphism. In the third part we study the product of Curtis homomorphisms and prove, amongst other things, Theorems 3.7 and 3.13 stated above. In the last part, we study the Curtis-Shoji homomorphism and prove statement (c) of Conjecture 2 when ℓ does not divide $|W|$.

Notation - If A is a finite dimensional algebra over a field, we denote by $\mathcal{R}(A)$ the Grothendieck group of the category of finitely generated A-modules. If M is a finitely generated A-module, we denote by $[M]$ its class in $\mathcal{R}(A)$. The opposite algebra of A will be denoted by A°.

All along this paper, we fix a prime number p, an algebraic closure $\mathbb{F}$ of the finite field with p elements $\mathbb{F}_p$, a prime number ℓ different from p and an algebraic extension K of the ℓ-adic field $\mathbb{Q}_\ell$. Let $\mathcal{O}$ be the ring of integers of K, let $\mathfrak{l}$ be the maximal ideal of $\mathcal{O}$ and let k denote the residue field of $\mathcal{O}$: k is an algebraic extension of the finite field $\mathbb{F}_\ell$. Throughout this paper, we assume that the ℓ-modular system $(K, \mathcal{O}, k)$ is sufficiently large for all the finite groups considered in this paper.

If Λ is a commutative $\mathcal{O}$-algebra (for instance $\Lambda = k$ or K), and if M is an $\mathcal{O}$-module, we set $\Lambda M = \Lambda \otimes_{\mathcal{O}} M$. If $f : M \to N$ is a morphism of $\mathcal{O}$-modules, we define ${}_\Lambda f : \Lambda M \to \Lambda N$ to be the morphism $\mathrm{Id}_\Lambda \otimes_{\mathcal{O}} f$. If V is a free left Λ-module, we denote by $V^* = \mathrm{Hom}_\Lambda(V, \Lambda)$ its dual: if V is a left A-module for some Λ-algebra A, then V^* is seen as a right A-module.

If G is a finite group, we denote by $\mathrm{Irr}\, G$ the set of irreducible characters of G over K. If $\chi \in \mathrm{Irr}\, G$, let e_χ (or e_χ^G if we need to emphasize the ambient group) denote the associated central primitive idempotent of KG :

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) g.$$

The conjugacy relation in G is denoted by $\sim$ or $\sim_G$ if necessary.

1. Background material

1.A. **The set-up.** We fix once and for all a connected reductive algebraic group $\mathbf{G}$ over $\mathbb{F}$ and we assume that it is endowed with an isogeny $F : \mathbf{G} \to \mathbf{G}$ such that

some power of F is a Frobenius endomorphism of $\mathbf{G}$ with respect to some rational structure on $\mathbf{G}$ over a finite extension of $\mathbb{F}_p$. We denote by q the positive real number such that, for every $\delta \geqslant 1$ such that F^δ is a Frobenius endomorphism of $\mathbf{G}$ over a finite field with r elements, we have $r = q^\delta$.

1.B. **Gelfand-Graev representations.** We fix an F-stable Borel subgroup $\mathbf{B}_\mathbf{G}$ of $\mathbf{G}$ and an F-stable maximal torus of $\mathbf{B}_\mathbf{G}$. Let $\mathbf{U}_\mathbf{G}$ denote the unipotent radical of $\mathbf{B}_\mathbf{G}$. We fix once and for all a *regular* linear character $\psi : \mathbf{U}_\mathbf{G}^F \to \mathcal{O}^\times \subset K^\times$ (in the sense of [DLM1, Definition 2.3]). Since $|\mathbf{U}_\mathbf{G}^F| = q^{\dim \mathbf{U}_\mathbf{G}}$ is a power of p, the primitive central idempotent e_ψ of $K\mathbf{U}_\mathbf{G}^F$ belongs to $\mathcal{O}\mathbf{U}_\mathbf{G}^F$. We denote by $\mathcal{O}_\psi$ the projective $\mathcal{O}\mathbf{U}_\mathbf{G}^F$-module $\mathcal{O}\mathbf{U}_\mathbf{G}^F e_\psi$: it is $\mathcal{O}$-free of rank one and is acted on by $\mathbf{U}_\mathbf{G}^F$ through ψ. Let

$$\Gamma^\mathbf{G} = \mathcal{O}\mathbf{G}^F e_\psi \simeq \mathrm{Ind}_{\mathbf{U}_\mathbf{G}^F}^{\mathbf{G}^F} \mathcal{O}_\psi.$$

Then $\Gamma^\mathbf{G}$ is a projective $\mathcal{O}\mathbf{G}^F$-module; the corresponding representation is called a *Gelfand-Graev representation* of $\mathbf{G}^F$.

Let $\mathcal{H}^\mathbf{G}$ denote the endomorphism algebra of the $\mathcal{O}\mathbf{G}^F$-module $\Gamma^\mathbf{G}$. We have

$$\mathcal{H}^\mathbf{G} \simeq (e_\psi \mathcal{O}\mathbf{G}^F e_\psi)^\circ \tag{1.1}$$

Since $\mathcal{O}\mathbf{G}^F$ is a symmetric algebra, we have that

(1.2) $\mathcal{H}^\mathbf{G}$ *is symmetric.*

The next result is much more difficult (see [S, Theorem 15] for the general case):

Theorem 1.3. *The algebra $\mathcal{H}^\mathbf{G}$ is commutative.*

Therefore, we shall identify the algebras $\mathcal{H}^\mathbf{G}$ and $e_\psi \mathcal{O}\mathbf{G}^F e_\psi$.

If Λ is a commutative $\mathcal{O}$-algebra, we denote by e_ψ^Λ the idempotent $1_\Lambda \otimes_\mathcal{O} e_\psi$ of $\Lambda\mathbf{U}_\mathbf{G}^F = \Lambda \otimes_\mathcal{O} \mathcal{O}\mathbf{U}_\mathbf{G}^F$. Since $\Gamma^\mathbf{G}$ is projective, the $\Lambda\mathbf{G}^F$-module $\Lambda\Gamma^\mathbf{G}$ is also projective and its endomorphism algebra is $\Lambda\mathcal{H}^\mathbf{G}$ (since it is isomorphic to $\mathrm{Hom}_\Lambda(\Lambda\Gamma^\mathbf{G}, \Lambda) \otimes_{\Lambda\mathbf{G}^F} \Lambda\Gamma^\mathbf{G}$). We have of course (taking into account that $\mathcal{H}^\mathbf{G}$ is symmetric)

$$\Lambda\mathcal{H}^\mathbf{G} = e_\psi^\Lambda \Lambda\mathbf{G}^F e_\psi^\Lambda. \tag{1.4}$$

Since $K\mathbf{G}^F$ is split semisimple,

(1.5) *The algebra $K\mathcal{H}^\mathbf{G}$ is split semisimple.*

Remark 1.6 - There might be several Gelfand-Graev representations of $\mathcal{O}\mathbf{G}^F$. But they are all conjugate by elements $g \in \mathbf{G}$ such $g^{-1}F(g)$ belongs to the centre of $\mathbf{G}$, and this gives a parametrization of Gelfand-Graev representations by the group of F-conjugacy classes in the centre of $\mathbf{G}$ (see [DLM1, 2.4.10]). In particular, their endomorphism algebras are all isomorphic.

Moreover, if the centre of $\mathbf{G}$ is connected, there is only one (up to isomorphism) Gelfand-Graev representation. In special orthogonal or symplectic groups in odd characteristic, there are two (isomorphism classes of) Gelfand-Graev representations. □

1.C. **Representations of $K\mathcal{H}^{\mathbf{G}}$.** Let $(\mathbf{G}^*, F^*)$ be a dual pair to $(\mathbf{G}, F)$ in the sense of Deligne and Lusztig [DL, Definition 5.21]. We denote by $\mathbf{G}^*_{\mathrm{sem}}$ the set of semisimple elements of $\mathbf{G}^*$. If $s \in \mathbf{G}^{*F^*}_{\mathrm{sem}}$, we denote by $(s)_{\mathbf{G}^{*F^*}}$ its conjugacy class in $\mathbf{G}^{*F^*}$ and by $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$ the associated rational Lusztig series (see [DM, Page 136]). We denote by $\chi_s^{\mathbf{G}}$ the unique element of $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$ which is an irreducible component of the character afforded by $K\Gamma^{\mathbf{G}}$. We view it as a function $K\mathbf{G}^F \to K$ and we denote by $\chi_s^{\mathcal{H}^{\mathbf{G}}}$ its restriction to $K\mathcal{H}^{\mathbf{G}}$. Then (see [DL, Theorem 10.7] for the case where the centre of $\mathbf{G}$ is connected and [A] for the general case; see also [B3, Remark of Page 80] for the case where F is not a Frobenius endomorphism),

$$[K\Gamma^{\mathbf{G}}] = \sum_{(s)_{\mathbf{G}^{*F^*}} \in \mathbf{G}^{*F^*}_{\mathrm{sem}}/\sim} \chi_s^{\mathbf{G}}. \tag{1.7}$$

Therefore, the next proposition is a particular case of [CR, Theorem 11.25 and Corollaries 11.26 and 11.27], taking into account that $K\mathcal{H}^{\mathbf{G}}$ is semisimple and commutative (or, equivalently, that $K\Gamma^{\mathbf{G}}$ is multiplicity free):

Proposition 1.8. *We have:*

(a) *The map $s \mapsto \chi_s^{\mathcal{H}^{\mathbf{G}}}$ induces a bijection between the set of $\mathbf{G}^{*F^*}$-conjugacy classes of semisimple elements of $\mathbf{G}^{*F^*}$ and the set of irreducible characters of $K\mathcal{H}^{\mathbf{G}}$.*

(b) *The map $s \mapsto e_{\chi_s^{\mathbf{G}}} e_\psi$ induces a bijection between the set of $\mathbf{G}^{*F^*}$-conjugacy classes of semisimple elements of $\mathbf{G}^{*F^*}$ and the set of primitive idempotents of $K\mathcal{H}^{\mathbf{G}}$.*

Moreover, if $s \in \mathbf{G}^{*F^*}_{\mathrm{sem}}$, then:

(c) *We have $\chi_s^{\mathcal{H}^{\mathbf{G}}}(e_{\chi_s^{\mathbf{G}}} e_\psi) = 1$ and $\chi_s^{\mathcal{H}^{\mathbf{G}}}(e_{\chi_t^{\mathbf{G}}} e_\psi) = 0$ if $t \in \mathbf{G}^{*F^*}_{\mathrm{sem}}$ is not conjugate to s in $\mathbf{G}^{*F^*}$.*

(d) *$e_{\chi_s^{\mathbf{G}}} e_\psi$ is the primitive idempotent of $K\mathcal{H}^{\mathbf{G}}$ associated with the irreducible character $\chi_s^{\mathcal{H}^{\mathbf{G}}}$.*

(e) *The $K\mathbf{G}^F$-module $K\mathbf{G}^F e_{\chi_s^{\mathbf{G}}} e_\psi$ is irreducible and affords the character $\chi_s^{\mathbf{G}}$.*

(f) *If $\chi \in \mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$ and if $\chi \neq \chi_s^{\mathbf{G}}$, then $\chi(h) = 0$ for all $h \in K\mathcal{H}^{\mathbf{G}}$.*

Since $K\mathcal{H}^{\mathbf{G}}$ is split and commutative, all its irreducible representations have dimension one. In other words, all its irreducible characters are morphisms of K-algebras $K\mathcal{H}^{\mathbf{G}} \to K$. So, as a consequence of the Proposition 1.8, we get that the

map

(1.9)
$$\begin{array}{rccc} \chi^{\mathcal{H}^{\mathbf{G}}} : & K\mathcal{H}^{\mathbf{G}} & \longrightarrow & \displaystyle\prod_{(s)_{\mathbf{G}^{*F^*}} \in \mathbf{G}^{*F^*}_{\mathrm{sem}}/\sim} K \\ & h & \longmapsto & (\chi_s^{\mathcal{H}^{\mathbf{G}}}(h))_{(s)_{\mathbf{G}^{*F^*}} \in \mathbf{G}^{*F^*}_{\mathrm{sem}}/\sim} \end{array}$$

is an isomorphism of K-algebras. It corresponds to the decomposition

(1.10)
$$K\mathcal{H}^{\mathbf{G}} = \bigoplus_{(s)_{\mathbf{G}^{*F^*}} \in \mathbf{G}^{*F^*}_{\mathrm{sem}}/\sim} K\mathcal{H}^{\mathbf{G}} e_{\chi_s^{\mathbf{G}}} e_\psi.$$

2. A generalization of the Curtis homomorphisms

In [C, Theorem 4.2], Curtis constructed a homomorphism of algebras $f_{\mathbf{T}} : \mathcal{H}^{\mathbf{G}} \to \mathcal{O}\mathbf{T}^F$, for $\mathbf{T}$ an F-stable maximal torus of $\mathbf{G}$ (in fact, Curtis constructed a homomorphism of algebras $K\mathcal{H}^{\mathbf{G}} \to K\mathbf{T}^F$ but it is readily checked from his formulas that it is defined over $\mathcal{O}$). We propose here a generalization of this construction to the case where $\mathbf{T}$ is replaced by an F-stable Levi subgroup $\mathbf{L}$ of a parabolic subgroup of $\mathbf{G}$: we then get a morphism $K\mathcal{H}^{\mathbf{G}} \to K\mathcal{H}^{\mathbf{L}}$ (note that, if $\mathbf{L}$ is a maximal torus, then $\mathcal{H}^{\mathbf{L}} = \mathcal{O}\mathbf{T}^F$). We conjecture that this morphism is defined over $\mathcal{O}$ and prove it whenever $\mathcal{G}^+(\mathbf{G}, \mathbf{L}, \mathbf{P})$ holds or whenever $\mathbf{L}$ is a maximal torus (see Theorem 2.7) or whenever ℓ does not divide the order of W (see Corollary 3.12).

2.A. **A morphism $K\mathcal{H}^{\mathbf{G}} \to K\mathcal{H}^{\mathbf{L}}$.** Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and assume that $\mathbf{P}$ admits an F-stable Levi complement $\mathbf{L}$.

The Gelfand-Graev representation $\Gamma^{\mathbf{G}}$ of $\mathcal{O}\mathbf{G}^F$ having been fixed, there is a well-defined (up to isomorphism) Gelfand-Graev representation $\Gamma^{\mathbf{L}}$ of $\mathcal{O}\mathbf{L}^F$ associated to it [B3, Page 77] (see also [B1]). We fix an F-stable Borel subgroup $\mathbf{B}_{\mathbf{L}}$ of $\mathbf{L}$ and we denote by $\mathbf{U}_{\mathbf{L}}$ its unipotent radical. We fix once and for all a regular linear character $\psi_{\mathbf{L}}$ of $\mathbf{U}_{\mathbf{L}}^F$ such that $\Gamma^{\mathbf{L}} = \mathrm{Ind}_{\mathbf{U}_{\mathbf{L}}^F}^{\mathbf{L}^F} \mathcal{O}_{\psi_{\mathbf{L}}} = \mathcal{O}\mathbf{L}^F e_{\psi_{\mathbf{L}}}$. We identify $\mathcal{H}^{\mathbf{L}}$ with $e_{\psi_{\mathbf{L}}}\mathcal{O}\mathbf{L}^F e_{\psi_{\mathbf{L}}}$. We also fix an F^*-stable Levi subgroup $\mathbf{L}^*$ of a parabolic subgroup of $\mathbf{G}^*$ dual to $\mathbf{L}$ (this is well-defined up to conjugacy by an element of $\mathbf{G}^{*F^*}$: see [DM, Page 113]). We then define ${}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}\, K\mathcal{H}^{\mathbf{G}} \to K\mathcal{H}^{\mathbf{L}}$ as the unique linear map such that, for any semisimple element $s \in \mathbf{G}^{*F^*}$,

$${}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(e_{\chi_s^{\mathbf{G}}} e_\psi) = \sum_{\substack{(t)_{\mathbf{L}^{*F^*}} \in \mathbf{L}^{*F^*}_{\mathrm{sem}}/\sim \\ t \in (s)_{\mathbf{G}^{*F^*}}}} e_{\chi_t^{\mathbf{L}}} e_{\psi_{\mathbf{L}}}.$$

Note that this does not depend on the choice of the representative s in its conjugacy class.

Proposition 2.1. *The map* ${}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}$ *is an homomorphism of algebras. Moreover, if* $s \in \mathbf{L}_{\mathrm{sem}}^{*F^*}$, *then*

$$\chi_s^{\mathcal{H}^{\mathbf{L}}} \circ {}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}} = \chi_s^{\mathcal{H}^{\mathbf{G}}}.$$

Proof. Since the image of an idempotent is an idempotent (and since $K\mathcal{H}^{\mathbf{G}}$ and $K\mathcal{H}^{\mathbf{L}}$ are split semisimple and commutative), we get the first statement. The second is obtained by applying both sides to each primitive idempotent $e_{\chi_t^{\mathbf{G}}} e_\psi$ of $K\mathcal{H}^{\mathbf{G}}$ ($t \in \mathbf{G}_{\mathrm{sem}}^{*F^*}$). $\square$

Another easy consequence of the definition is the following

Proposition 2.2. *If* $\mathbf{M}$ *is an* F*-stable Levi subgroup of a parabolic subgroup of* $\mathbf{G}$ *and if* $\mathbf{L} \subset \mathbf{M}$, *then* ${}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{M}} \circ {}_K\mathrm{Cur}_{\mathbf{M}}^{\mathbf{G}} = {}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}$.

2.B. **Deligne-Lusztig functors and Gelfand-Graev representations.** Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and assume that $\mathbf{P}$ admits an F-stable Levi complement $\mathbf{L}$. Let $\mathbf{V}$ denote the unipotent radical of $\mathbf{P}$. We set

$$\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g^{-1}F(g) \in \mathbf{V} \cdot F(\mathbf{V})\}$$

and $d_{\mathbf{P}} = \dim(\mathbf{V}) - \dim(\mathbf{V} \cap F(\mathbf{V}))$. Then $\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}}$ is a locally closed smooth variety of pure dimension $d_{\mathbf{P}}$. If $\Lambda = \mathcal{O}$, K or $\mathcal{O}/\mathfrak{l}^n$, the complex of cohomology with compact support of $\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}}$ with coefficients in Λ, which is denoted by $\mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}}, \Lambda)$, is a bounded complex of $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$-bimodules which is perfect as a complex of left $\Lambda\mathbf{G}^F$-modules and is also perfect as a complex of right $\Lambda\mathbf{L}^F$-modules (see [DL, §3.8]). Its i-th cohomology group is denoted by $H_c^i(\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}}, \Lambda)$: it is a $(\Lambda\mathbf{G}^F, \Lambda\mathbf{L}^F)$-bimodule. For $\Lambda = \mathcal{O}$, this complex induces two functors between bounded derived categories

$$\begin{array}{rccc} \mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : & D^b(\mathcal{O}\mathbf{L}^F) & \longrightarrow & D^b(\mathcal{O}\mathbf{G}^F) \\ & C & \longmapsto & \mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}}, \mathcal{O}) \otimes_{\mathcal{O}\mathbf{L}^F} C \end{array}$$

and

$$\begin{array}{rccc} {}^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : & D^b(\mathcal{O}\mathbf{G}^F) & \longrightarrow & D^b(\mathcal{O}\mathbf{L}^F) \\ & C & \longmapsto & \mathrm{R}\,\mathrm{Hom}_{\mathcal{O}\mathbf{G}^F}^{\bullet}(\mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}}, \mathcal{O}), C). \end{array}$$

These functors are respectively called *Deligne-Lusztig induction* and *restriction*. By extending the scalars to K, they induce linear maps between the Grothendieck groups $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \mathcal{R}(K\mathbf{L}^F) \to \mathcal{R}(K\mathbf{G}^F)$ and ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : \mathcal{R}(K\mathbf{G}^F) \to \mathcal{R}(K\mathbf{L}^F)$. We have

$$R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}[M] = \sum_{i \geqslant 0} (-1)^i [H_c^i(\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}}, K) \otimes_{K\mathbf{L}^F} M]$$

and

$${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}[N] = \sum_{i \geqslant 0} (-1)^i [H_c^i(\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}}, K)^* \otimes_{K\mathbf{G}^F} N]$$

for all $K\mathbf{G}^F$-modules N and all $K\mathbf{L}^F$-modules M.

If $(g,l) \in K\mathbf{G}^F \times K\mathbf{L}^F$, we set

$$\mathrm{Tr}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(g,l) = \sum_{i\geqslant 0} (-1)^i \,\mathrm{Tr}((g,l), H^i_c(\mathbf{Y}^{\mathbf{G}}_{\mathbf{P}}, K)).$$

If $(g,l) \in \mathbf{G}^F \times \mathbf{L}^F$, then $\mathrm{Tr}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(g,l)$ is a rational integer which does not depend on the prime number ℓ (see [DL, Proposition 3.3]). If χ_M (respectively χ_N) denotes the character afforded by a $K\mathbf{L}^F$-module M (respectively a $K\mathbf{G}^F$-module N), then the character afforded by the virtual module $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}[M]$ (respectively ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}[N]$) will be denoted by $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\chi_M$ (respectively ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\chi_N$): it satisfies

$$R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\chi_M(g) = \frac{1}{|\mathbf{L}^F|}\sum_{l\in\mathbf{L}^F} \mathrm{Tr}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(g,l)\chi_M(l^{-1})$$

(respectively $${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}\chi_N(l) = \frac{1}{|\mathbf{G}^F|}\sum_{g\in\mathbf{G}^F} \mathrm{Tr}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}(g,l)\chi_N(g^{-1}) \quad)$$

for all $g \in \mathbf{G}^F$ (respectively $l \in \mathbf{L}^F$).

COMMENTS (INDEPENDENCE ON THE PARABOLIC) - If $\mathbf{P}'$ is another parabolic subgroup of $\mathbf{G}$ having $\mathbf{L}$ as a Levi complement, then the Deligne-Lusztig varieties $\mathbf{Y}^{\mathbf{G}}_{\mathbf{P}}$ and $\mathbf{Y}^{\mathbf{G}}_{\mathbf{P}'}$ are in general non-isomorphic: they might even have different dimension (however, note that $(-1)^{d_{\mathbf{P}}} = (-1)^{d_{\mathbf{P}'}}$, i.e. $d_{\mathbf{P}} \equiv d_{\mathbf{P}'} \mod 2$). As a consequence, the Deligne-Lusztig functors $\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}$ and $\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}'}$ can be really different. However, it is conjectured in general that $R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}'}$ and ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = {}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}'}$. This is equivalent to say that $\mathrm{Tr}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = \mathrm{Tr}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}'}$.

For instance, we have $\mathrm{Tr}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}} = \mathrm{Tr}^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}'}$ if $\mathbf{L}$ is a maximal torus [DL, Corollary 4.3], or if $\mathbf{P}$ and $\mathbf{P}'$ are F-stable (this is due to Deligne: a proof can be found in [DM, Theorem 5.1]), or if F is a Frobenius endomorphism and $q \neq 2$ (see [BM]). In all these cases, this fact is a consequence of the Mackey formula for Deligne-Lusztig maps. □

The Gelfand-Graev representation $\Gamma^{\mathbf{L}}$ satisfies the following property:

Theorem 2.3. *Assume that one of the following three conditions is satisfied:*

(1) $\mathbf{P}$ *is* F*-stable.*
(2) *The centre of* $\mathbf{L}$ *is connected.*
(3) p *is almost good for* $\mathbf{G}$*,* F *is a Frobenius endomorphism of* $\mathbf{G}$ *and* q *is large enough.*

Then ${}^*R^{\mathbf{G}}_{\mathbf{L}\subset\mathbf{P}}[K\Gamma^{\mathbf{G}}] = (-1)^{d_{\mathbf{P}}}[K\Gamma^{\mathbf{L}}]$.

Proof. (1) is due to Rodier: a proof may be found in [DLM1, Theorem 2.9]. (2) is proved in [DLM1, Proposition 5.4]. For (3) see [DLM2, Theorem 3.7], [B2, Theorem 15.2] and [B3, Theorem 14.11]. □

It is conjectured that the above theorem holds without any restriction (on p, q, F or the centre of $\mathbf{L}$...). However, at the time of the writing of this paper, this general conjecture is still unproved. So we will denote by $\mathcal{G}(\mathbf{G},\mathbf{L},\mathbf{P})$ the property

$$(\mathcal{G}(\mathbf{G},\mathbf{L},\mathbf{P})) \qquad {}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}[K\Gamma^{\mathbf{G}}] = (-1)^{d_{\mathbf{P}}}[K\Gamma^{\mathbf{L}}].$$

Most of the results of this subsection will be valid only under the hypothesis that $\mathcal{G}(\mathbf{G},\mathbf{L},\mathbf{P})$ holds. In light of the above theorem, and as there are many other indications that $\mathcal{G}(\mathbf{G},\mathbf{L},\mathbf{P})$ holds in general, this should not be viewed as a big restriction.

In fact, there is also strong evidence that the perfect complex of $\mathcal{O}\mathbf{L}^F$-modules ${}^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\Gamma^{\mathbf{G}}$ is concentrated in degree $d_{\mathbf{P}}$: more precisely, it is conjectured [BR2, Conjecture 2.3] that

$$(\mathcal{G}^+(\mathbf{G},\mathbf{L},\mathbf{P})) \qquad {}^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\Gamma^{\mathbf{G}} \simeq \Gamma^{\mathbf{L}}[-d_{\mathbf{P}}]$$

The conjectural property $\mathcal{G}^+(\mathbf{G},\mathbf{L},\mathbf{P})$ is a far reaching extension of $\mathcal{G}(\mathbf{G},\mathbf{L},\mathbf{P})$. It is known to hold only if $\mathbf{P}$ is F-stable (see Theorem 2.3 (1) or [BR2, Theorem 2.1] for a module-theoretic proof) or if $\mathbf{L}$ is a maximal torus and $(\mathbf{P},F(\mathbf{P}))$ lies in the orbit associated with an element of the Weyl group which is a product of simple reflections lying in different F-orbits [BR2, Theorem 3.10]. Of course, a proof of this conjecture would produce immediately a morphism of $\mathcal{O}$-algebras $\mathcal{H}^{\mathbf{G}} \to \mathcal{H}^{\mathbf{L}}$ (which is uniquely determined since $\mathcal{H}^{\mathbf{L}}$ is commutative). However, as we shall see in this section, we only need that $\mathcal{G}(\mathbf{G},\mathbf{L},\mathbf{P})$ holds to get the following result:

Proposition 2.4. *If $\mathcal{G}(\mathbf{G},\mathbf{L},\mathbf{P})$ holds, then, for all $h \in K\mathcal{H}^{\mathbf{G}} \subset K\mathbf{G}^F$,*

$${}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(h) = (-1)^{d_{\mathbf{P}}} \sum_{s \in \mathbf{L}_{\mathrm{sem}}^{*F^*}/\sim_{\mathbf{L}^{*F^*}}} \mathrm{Tr}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(h, e_{\chi_s^{\mathbf{L}}} e_{\psi_{\mathbf{L}}}) e_{\chi_s^{\mathbf{L}}} e_{\psi_{\mathbf{L}}}.$$

Proof. We assume throughout this proof that $\mathcal{G}(\mathbf{G},\mathbf{L},\mathbf{P})$ holds. We denote by $\boldsymbol{\Gamma}^{\mathbf{G}}$ the character afforded by the module $K\Gamma^{\mathbf{G}}$. Let $f : K\mathcal{H}^{\mathbf{G}} \to K\mathcal{H}^{\mathbf{L}}$ be the map defined by

$$f(h) = (-1)^{d_{\mathbf{P}}} \sum_{s \in \mathbf{L}_{\mathrm{sem}}^{*F^*}/\sim_{\mathbf{L}^{*F^*}}} \mathrm{Tr}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(h, e_{\chi_s^{\mathbf{L}}} e_{\psi_{\mathbf{L}}}) e_{\chi_s^{\mathbf{L}}} e_{\psi_{\mathbf{L}}}.$$

Let $s \in \mathbf{L}_{\mathrm{sem}}^{*F^*}$. In order to prove the proposition, we only need to check that

$$(?) \qquad \chi_s^{\mathcal{H}^{\mathbf{L}}} \circ f = \chi_s^{\mathcal{H}^{\mathbf{L}}} \circ {}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}.$$

First, note that $\langle \chi_s^{\mathbf{L}}, \Gamma^{\mathbf{L}} \rangle_{\mathbf{L}^F} = 1$ so, by adjunction, and since $\mathcal{G}(\mathbf{G},\mathbf{L},\mathbf{P})$ holds, we have $\langle R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \chi_s^{\mathbf{L}}, \Gamma^{\mathbf{G}} \rangle_{\mathbf{G}^F} = (-1)^{d_{\mathbf{P}}}$. Since $\chi_s^{\mathbf{G}}$ is the unique irreducible constituent of $\boldsymbol{\Gamma}^{\mathbf{G}}$ lying in $\mathcal{E}(\mathbf{G}^F, (s)_{\Gamma^{*F^*}})$ and since all the irreducible constituents of $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} \chi_s^{\mathbf{L}}$

belong to $\mathcal{E}(\mathbf{G}^F, (s)_{\mathbf{G}^{*F^*}})$ (see for instance [B3, Theorem 11.10]), we have

$$R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\chi_s^{\mathbf{L}} = (-1)^{d_{\mathbf{P}}}\chi_s^{\mathbf{G}} + \sum_{\substack{\chi\in\mathcal{E}(\mathbf{G}^F,(s)_{\mathbf{G}^{*F^*}})\\ \chi\neq\chi_s^{\mathbf{G}}}} m_\chi\chi$$

for some $m_\chi \in \mathbb{Z}$. By Proposition 1.8 (f), we have

$$(*)\qquad \big(R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\chi_s^{\mathbf{L}}\big)(h) = (-1)^{d_{\mathbf{P}}}\chi_s^{\mathbf{G}}(h) = (-1)^{d_{\mathbf{P}}}\chi_s^{\mathcal{H}^{\mathbf{G}}}(h) = \chi_s^{\mathcal{H}^{\mathbf{L}}}({}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(h))$$

for all $h \in K\mathcal{H}^{\mathbf{G}}$. On the other hand, we have

$$\chi_s^{\mathcal{H}^{\mathbf{L}}}(f(h)) = (-1)^{d_{\mathbf{P}}}\,\mathrm{Tr}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(h, e_{\chi_s^{\mathbf{L}}}e_{\psi_{\mathbf{L}}}).$$

But, since the actions of h and of $e_{\chi_s^{\mathbf{L}}}e_{\psi_{\mathbf{L}}}$ on the cohomology groups $H_c^i(\mathbf{Y}_{\mathbf{P}}, K)$ commute and since $e_{\chi_s^{\mathbf{L}}}e_{\psi_{\mathbf{L}}}$ is an idempotent, we have that $\mathrm{Tr}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(h, e_{\chi_s^{\mathbf{L}}}e_{\psi_{\mathbf{L}}})$ is the trace of h on the virtual module

$$\sum_{i\geqslant 0}(-1)^i[H_c^i(\mathbf{Y}_{\mathbf{P}}, K)e_{\chi_s^{\mathbf{L}}}e_{\psi_{\mathbf{L}}})] = \sum_{i\geqslant 0}(-1)^i[H_c^i(\mathbf{Y}_{\mathbf{P}}, K)\otimes_{K\mathbf{L}^F} K\mathbf{L}^F e_{\chi_s^{\mathbf{L}}}e_{\psi_{\mathbf{L}}})].$$

Now, by Proposition 1.8 (e), the $K\mathbf{L}^F$-module $K\mathbf{L}^F e_{\chi_s^{\mathbf{L}}}e_{\psi_{\mathbf{L}}}$ affords the character $\chi_s^{\mathbf{L}}$. So it follows that

$$(**)\qquad \mathrm{Tr}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(h, e_{\chi_s^{\mathbf{L}}}e_{\psi_{\mathbf{L}}}) = \big(R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\chi_s^{\mathbf{L}}\big)(h).$$

So, (?) follows from the comparison of $(*)$ and $(**)$. □

Proposition 2.5. *If* $\mathbf{B}$ *is a Borel subgroup of* $\mathbf{G}$ *and if* $\mathbf{T}$ *is a maximal torus of* $\mathbf{B}$, *then* $\mathcal{G}(\mathbf{G}, \mathbf{T}, \mathbf{B})$ *holds and* ${}_K\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}$ *coincides with Curtis homomorphism* $f_{\mathbf{T}}$ *defined in* [C, Theorem 4.2]. *We have, for all* $h \in K\mathcal{H}^{\mathbf{G}}$,

$${}_K\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}(h) = \frac{1}{|\mathbf{T}^F|}\sum_{t\in\mathbf{T}^F}\mathrm{Tr}_{\mathbf{T}\subset\mathbf{B}}^{\mathbf{G}}(h, t)t^{-1}.$$

Remark - The formula given in Proposition 2.5 gives a concise form for Curtis homomorphism. It can be checked directly, using the character formula [DM, Proposition 12.2], that this indeed coincides with the formulas given by Curtis in terms of Green functions [C, 4.3]. However, we shall give a more theoretical proof of this coincidence. □

Proof. Since the centre of $\mathbf{T}$ is connected, $\mathcal{G}(\mathbf{G}, \mathbf{T}, \mathbf{B})$ holds by Theorem 2.3 (2). Also, $\mathbf{U}_{\mathbf{T}} = 1$, $\psi_{\mathbf{T}} = 1$, so $K\mathcal{H}^{\mathbf{T}} = K\mathbf{T}^F$. So the primitive idempotents of $K\mathcal{H}^{\mathbf{T}}$ are the primitive idempotents of $K\mathbf{T}^F$ and the formula given above can be obtained by a straightforward computation.

Now, let $\mathbf{T}^*$ be an F^*-stable maximal torus of $\mathbf{G}^*$ dual to $\mathbf{T}$. If $s \in \mathbf{T}^{*F^*}$, then $\chi_s^{\mathbf{T}} = \chi_s^{\mathcal{H}^{\mathbf{T}}}$ and, by [C, Theorem 4.2], Curtis homomorphism $f_{\mathbf{T}} : K\mathcal{H}^{\mathbf{G}} \to K\mathbf{T}^F$ satisfies

$$\chi_s^{\mathbf{T}} \circ f_{\mathbf{T}} = \chi_s^{\mathcal{H}^{\mathbf{G}}}.$$

Since $\chi^{\mathcal{H}^{\mathbf{T}}}$ is an isomorphism of K-algebras, we get from Proposition 2.1 that $f_{\mathbf{T}} = \mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}$. $\square$

REMARK 2.6 - If χ is a class function on $\mathbf{L}^F$ (which can be seen as a class function on $K\mathbf{L}^F$) and if $\mathcal{G}(\mathbf{G},\mathbf{L},\mathbf{P})$ holds, then we have

$$\chi(\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(h)) = (-1)^{d_{\mathbf{P}}} R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(\chi)(h)$$

for all $h \in K\mathcal{H}^{\mathbf{G}}$. For this, one may assume that $\chi \in \mathrm{Irr}\,\mathbf{L}^F$. Let $s \in \mathbf{L}_{\mathrm{sem}}^{*F^*}$ be such that $\chi \in \mathcal{E}(\mathbf{L}^F, (s)_{\mathbf{L}^{*F^*}})$. If $\chi = \chi_s^{\mathbf{L}}$, then this is the equality $(*)$ in the proof of the Proposition 2.4. If $\chi \neq \chi_s^{\mathbf{L}}$, we must show that $R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(\chi)(h) = 0$ for all $h \in K\mathcal{H}^{\mathbf{G}}$ (see Proposition 1.8 (f)). Let $\gamma \in \mathrm{Irr}\,\mathbf{G}^F$ be such that $\langle \gamma, R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\chi\rangle_{\mathbf{G}^F} \neq 0$. Then $\gamma \in \mathcal{E}(\mathbf{G}^F(s)_{\mathbf{G}^{*F^*}})$ (see for instance [B3, Theorem 11.10]) and, by Proposition 1.8 (f), it is sufficient to show that $\gamma \neq \chi_s^{\mathbf{G}}$. But

$$\langle \chi_s^{\mathbf{G}}, R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\chi\rangle_{\mathbf{G}^F} = \langle \Gamma^{\mathbf{G}}, R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\chi\rangle_{\mathbf{G}^F} = \langle \Gamma^{\mathbf{L}}, \chi\rangle_{\mathbf{L}^F} = 0.$$

This shows the result. $\square$

2.C. **A morphism $\mathcal{H}^{\mathbf{G}} \to \mathcal{H}^{\mathbf{L}}$.** We conjecture that, in general, ${}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(\mathcal{H}^{\mathbf{G}}) \subset \mathcal{H}^{\mathbf{L}}$. At this stage of the paper, we are only able to prove it in the following cases (in Corollary 3.12, we shall see that this property also holds if ℓ does not divide the order of W):

Theorem 2.7. *We have:*

(a) *If $\mathcal{G}^+(\mathbf{G},\mathbf{L},\mathbf{P})$ holds, then ${}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(\mathcal{H}^{\mathbf{G}}) \subset \mathcal{H}^{\mathbf{L}}$ and the resulting morphism of $\mathcal{O}$-algebra $\mathcal{H}^{\mathbf{G}} \to \mathcal{H}^{\mathbf{L}}$ coincides with the functorial morphism coming from the isomorphism ${}^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\Gamma^{\mathbf{G}} \simeq \Gamma^{\mathbf{L}}[-d_{\mathbf{P}}]$.*

(b) *If $\mathbf{L}$ is a maximal torus, then ${}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(\mathcal{H}^{\mathbf{G}}) \subset \mathcal{H}^{\mathbf{L}}$.*

Proof. (b) follows easily from Proposition 2.5 and from the well-known fact that, if $(g,l) \in \mathbf{G}^F \times \mathbf{L}^F$, then $|\mathbf{L}^F|$ divides $\mathrm{Tr}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}(g,l)$ because $\mathbf{L}$ is a maximal torus.

(a) The complex $\mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}},\mathcal{O})$ is perfect as a complex of left $\mathcal{O}\mathbf{G}^F$-modules. Therefore, we have ${}^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}C = \mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}},\mathcal{O})^* \otimes_{\mathcal{O}\mathbf{G}^F} C$ for any complex C of $\mathcal{O}\mathbf{G}^F$-modules. If $\mathcal{G}^+(\mathbf{G},\mathbf{L},\mathbf{P})$ holds, then this means that we have an isomorphism $\mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}},\mathcal{O})^* e_\psi \simeq \Gamma^{\mathbf{L}}[-d_{\mathbf{P}}]$. In particular, the complex $\mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}},\mathcal{O})^* e_\psi$ is concentrated in degree $d_{\mathbf{P}}$. Therefore, there exists an $(\mathcal{O}\mathbf{L}^F,\mathcal{H}^{\mathbf{G}})$-bimodule P such that $\mathrm{R}\Gamma_c(\mathbf{Y}_{\mathbf{P}}^{\mathbf{G}},\mathcal{O})^* e_\psi \simeq P[-d_{\mathbf{P}}]$. Moreover, as a left $\mathcal{O}\mathbf{L}^F$-module, we have an isomorphism $\alpha : \mathcal{O}\mathbf{L}^F e_{\psi_{\mathbf{L}}} \xrightarrow{\sim} P$.

This induces a morphism

$$\begin{array}{rccc} \tilde{\alpha}: & \mathcal{H}^{\mathbf{G}} & \longrightarrow & \mathcal{H}^{\mathbf{L}} \\ & h & \longmapsto & \alpha^{-1}(\alpha(e_{\psi_{\mathbf{L}}})h). \end{array}$$

The morphism $\tilde{\alpha} : \mathcal{H}^{\mathbf{G}} \to \mathcal{H}^{\mathbf{L}}$ does not depend on the choice of α because $\mathcal{H}^{\mathbf{L}}$ is commutative. This morphism can be extended to a morphism ${}_K\tilde{\alpha} : K\mathcal{H}^{\mathbf{G}} \to K\mathcal{H}^{\mathbf{L}}$, $h \mapsto {}_K\alpha^{-1}({}_K\alpha(e_{\psi_{\mathbf{L}}})h)$. Now the Theorem would follow if we show that ${}_K\tilde{\alpha} = {}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}$.

So let $s \in \mathbf{G}^{*F^*}_{\mathrm{sem}}$. Let $\mathcal{E}$ be a set of representatives of $\mathbf{L}^F$-conjugacy classes which are contained in $\mathbf{L}^{*F^*} \cap (s)_{\mathbf{G}^{*F^*}}$ and let $e = \sum_{t\in\mathcal{E}} e_{\psi_{\mathbf{L}}} e_{\chi_t^{\mathbf{L}}}$. Then ${}^*R_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}\chi_s^{\mathbf{G}} = (-1)^{d_{\mathbf{P}}} \sum_{t\in\mathcal{E}} \chi_t^{\mathbf{L}}$. In particular, α induces an isomorphism

$$KPe_{\chi_s^{\mathbf{L}}} \simeq K\mathbf{L}^F e.$$

So this shows that ${}_K\tilde{\alpha}(e_\psi e_{\chi_s^{\mathbf{G}}}) = e$, as desired. $\square$

2.D. **Truncation at unipotent blocks.** We denote by $b^{\mathbf{G}}$ the sum of the *unipotent* block idempotents of $\mathbf{G}^F$. In other words,

$$b^{\mathbf{G}} = \sum_{\substack{s\in\mathbf{G}^{*F^*}_{\mathrm{sem}}/\sim \\ s \text{ is an } \ell\text{-element}}} \sum_{\chi\in\mathcal{E}(\mathbf{G}^F,(s)_{\mathbf{G}^{*F^*}})} e_\chi.$$

The algebra $\mathcal{H}^{\mathbf{G}}$ is a module over the centre of the $\mathcal{O}$-algebra $\mathcal{O}\mathbf{G}^F$: so $b^{\mathbf{G}}\mathcal{H}^{\mathbf{G}}$ is an $\mathcal{O}$-algebra with unit $b^{\mathbf{G}}e_\psi$. Note that

$$b^{\mathbf{G}}e_\psi = \sum_{\substack{s\in\mathbf{G}^{*F^*}_{\mathrm{sem}}/\sim \\ s \text{ is an } \ell\text{-element}}} e_{\chi_s^{\mathbf{G}}} e_\psi.$$

Now, by definition, we get

$${}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(b^{\mathbf{G}}e_\psi) = b^{\mathbf{L}}e_{\psi_{\mathbf{L}}}.$$

In particular,

$$\textbf{(2.8)} \qquad {}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(b^{\mathbf{G}}K\mathcal{H}^{\mathbf{G}}) \subset b^{\mathbf{L}}K\mathcal{H}^{\mathbf{L}}.$$

Let us also recall for future reference the following classical fact:

Proposition 2.9. *The projective $\mathcal{O}\mathbf{G}^F$-module $b^{\mathbf{G}}\Gamma^{\mathbf{G}}$ is indecomposable.*

Proof. See [CE2, Proposition 19.6 (i)]. Note that the statement in [CE2] is made under the hypotheses that $\mathbf{G}$ has connected center, but the proof applies without change in the general situation. $\square$

Corollary 2.10. *The algebra $b^{\mathbf{G}}\mathcal{H}^{\mathbf{G}}$ is local.*

3. Glueing Curtis homomorphisms for maximal tori

If $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$ and if $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{B}$, we then write $R_{\mathbf{T}}^{\mathbf{G}}$, and $\mathrm{Tr}_{\mathbf{T}}^{\mathbf{G}}$ for the maps $R_{\mathbf{T}\subset\mathbf{B}}^{\mathbf{G}}$ and $\mathrm{Tr}_{\mathbf{T}\subset\mathbf{B}}^{\mathbf{G}}$ (see the comments at the end of subsection 2.B and Proposition 2.1 (c)).

Let $\mathbf{T}_{\mathbf{G}}$ denote an F-stable maximal torus of $\mathbf{B}_{\mathbf{G}}$. We set $W = N_{\mathbf{G}}(\mathbf{T}_{\mathbf{G}})/\mathbf{T}_{\mathbf{G}}$. For each $w \in W$, we fix an element $g \in \mathbf{G}$ such that $g^{-1}F(g)$ belongs to $N_{\mathbf{G}}(\mathbf{T}_{\mathbf{G}})$ and represents w. We then set $\mathbf{T}_w = g\mathbf{T}_{\mathbf{G}}g^{-1}$. We then define, following [CS, Lemma 1],

$$\begin{array}{rrcl} \mathrm{Cur}^{\mathbf{G}} : & \mathcal{H}^{\mathbf{G}} & \longrightarrow & \displaystyle\prod_{w\in W} \mathcal{O}\mathbf{T}_w^F \\ & h & \longmapsto & \big(\mathrm{Cur}_{\mathbf{T}_w}^{\mathbf{G}}(h)\big)_{w\in W}. \end{array}$$

The aim of this section is to study the map $\mathrm{Cur}^{\mathbf{G}}$.

3.A. Properties of ${}_K\mathbf{Cur}^{\mathbf{G}}$.

Before studying $\mathrm{Cur}^{\mathbf{G}}$, we shall study the simpler map ${}_K\mathrm{Cur}^{\mathbf{G}}$. It turns out that ${}_K\mathrm{Cur}^{\mathbf{G}}$ is injective and it is relatively easy to describe its image: both facts were obtained by Curtis and Shoji [CS, Lemmas 1 and 5] but we shall present here a concise proof.

We first need to introduce some notation. If $w \in W$, we fix an F^*-stable maximal torus $\mathbf{T}_w^*$ dual to $\mathbf{T}_w$. If $t \in \mathbf{T}_w^{*F^*}$, then $\chi_t^{\mathbf{T}_w}$ is a linear character of $\mathbf{T}_w^F$. If s is a semisimple element of $\mathbf{G}^{*F^*}$, we set

$$e^{\mathbf{G}}(s) = \Big(\sum_{t\in(s)_{\mathbf{G}^{*F^*}}\cap\mathbf{T}_w^{*F^*}} e_{\chi_t^{\mathbf{T}_w}}\Big)_{w\in W} \in \prod_{w\in W} K\mathbf{T}_w^F.$$

Then, by definition, we have

$$\textbf{(3.1)} \qquad {}_K\mathrm{Cur}^{\mathbf{G}}(e_{\chi_s^{\mathbf{G}}}e_\psi) = e^{\mathbf{G}}(s).$$

Since $(e^{\mathbf{G}}(s))_{(s)\in\mathbf{G}^{*F^*}_{\mathrm{sem}}/\sim}$ is a K-linearly independent family in $\prod_{w\in W} K\mathbf{T}_w^F$, we get:

Proposition 3.2 (Curtis-Shoji)**.** *The map ${}_K\mathrm{Cur}^{\mathbf{G}}$ is injective and*

$$\mathrm{Im}\,{}_K\mathrm{Cur}^{\mathbf{G}} = \bigoplus_{(s)\in\mathbf{G}^{*F^*}_{\mathrm{sem}}/\sim} Ke^{\mathbf{G}}(s).$$

Corollary 3.3. *The map $\mathrm{Cur}^{\mathbf{G}}$ is injective.*

We shall now recall a characterization of elements of the image of ${}_K\mathrm{Cur}^{\mathbf{G}}$ which was obtained by Curtis and Shoji [CS, Lemma 5]. We need some notation. Let $\mathcal{S}^{\mathbf{G}}$ denote the set of pairs (w,θ) such that $w \in W$ and θ is a linear character of $\mathbf{T}_w^F$ (which may also be viewed as a morphism of algebras $\mathcal{O}\mathbf{T}_w^F \to \mathcal{O}$ or $K\mathbf{T}_w^F \to K$).

If (w,θ) and (w',θ') are two elements of $\mathcal{S}^{\mathbf{G}}$, we write $(w,\theta)\equiv(w',\theta')$ if $(\mathbf{T}_w,\theta)$ and $(\mathbf{T}_{w'},\theta')$ lie in the same rational series (see for instance [B3, Definition 9.4] for a definition).

Corollary 3.4 (Curtis-Shoji). *Let $t=(t_w)_{w\in W}\in\prod_{w\in W}K\mathbf{T}_w^F$. Then $t\in\operatorname{Im}{}_K\mathrm{Cur}^{\mathbf{G}}$ if and only if, for all (w,θ), $(w',\theta')\in\mathcal{S}^{\mathbf{G}}$ such that $(w,\theta)\equiv(w',\theta')$, we have $\theta(t_w)=\theta'(t_{w'})$.*

Proof. Let $t=(t_w)_{w\in W}\in\prod_{w\in W}K\mathbf{T}_w^F$. Since for all $w\in W$, $K\mathbf{T}_w^F$ is split commutative and semi-simple, the idempotents of $K\mathbf{T}_w^F$ form a K-basis of $K\mathbf{T}_w^F$, and we may write $t=\prod_{w\in W}\sum_{g\in\mathbf{T}_w^{*F^*}}\alpha_g^{\mathbf{T}_w}e_{\chi_g^{\mathbf{T}}}$, where $\alpha_g^{\mathbf{T}_w}\in K$.

Now, from Proposition 3.2 we have that $t\in\operatorname{Im}{}_K\mathrm{Cur}^{\mathbf{G}}$ if and only if, whenever g,g' are rationally conjugate semi-simple elements of $\mathbf{G}^{*F^*}$, then for any $w,w'\in W$ such that $g\in\mathbf{T}^{*F^*}$ and $g'\in\mathbf{T}'^{*F^*}$, we have $\alpha_g^{\mathbf{T}_w}=\alpha_{g'}^{\mathbf{T}_{w'}}$. On the other hand, if $g\in\mathbf{T}_w^{*F^*}$, then $\alpha_g^{\mathbf{T}_w}=\chi_g^{\mathbf{T}_w}(t_w)$. The result follows from the definition of the equivalence relation on $\mathcal{S}^{\mathbf{G}}$.

□

3.B. **Symmetrizing form.** The $\mathcal{O}$-algebra $\mathcal{H}^{\mathbf{G}}$ is symmetric. In particular, the $\mathcal{O}$-algebra $\operatorname{Im}\mathrm{Cur}^{\mathbf{G}}$ is symmetric (see Corollary 3.3). We shall give in this subsection a precise formula for the symmetrizing form on $\operatorname{Im}\mathrm{Cur}^{\mathbf{G}}$. For this, we introduce the following symmetrizing form

$$\begin{array}{rccc}\tilde{\tau}: & \displaystyle\prod_{w\in W}K\mathbf{T}_w^F & \longrightarrow & K\\ & (x_w)_{w\in W} & \longmapsto & \displaystyle\frac{1}{|W|}\sum_{w\in W}\tau_w(x_w),\end{array}$$

where $\tau_w:K\mathbf{T}_w^F\to K$ is the canonical symmetrizing form.

We denote by $\tau:\mathcal{O}\mathbf{G}^F\to\mathcal{O}$ the canonical symmetrizing form. We denote by $\tau_{\mathcal{H}}$ the restriction of $|\mathbf{U}_{\mathbf{G}}^F|\tau$ to $\mathcal{H}^{\mathbf{G}}$: it is a symmetrizing form on $\mathcal{H}^{\mathbf{G}}$ (recall that $|\mathbf{U}_{\mathbf{G}}^F|$ is invertible in $\mathcal{O}$ and is the highest power of p dividing $|\mathbf{G}^F|$). Note that

$$\tau_{\mathcal{H}}(e_\psi)=1.$$

Of course, the extension ${}_K\tau_{\mathcal{H}}:K\mathcal{H}^{\mathbf{G}}\to K$ is a symmetrizing form on $K\mathcal{H}^{\mathbf{G}}$. We have

(3.5) $$ {}_K\tau_{\mathcal{H}}=\tilde{\tau}\circ{}_K\mathrm{Cur}^{\mathbf{G}}.$$

Proof. Since τ_w is a class function on $\mathbf{T}_w^F$, we have, by Remark 2.6,

$$\tilde{\tau}({}_K\mathrm{Cur}^{\mathbf{G}}(h))=\frac{1}{|W|}\sum_{w\in W}R_{\mathbf{T}_w}^{\mathbf{G}}(\tau_w)(h)$$

for all $h \in K\mathcal{H}^{\mathbf{G}}$. But, by [DM, Proposition 12.9 and Corollary 12.14], we have

$$\frac{1}{|W|}\sum_{w\in W} R_{\mathbf{T}_w}^{\mathbf{G}}(\tau_w) = |\mathbf{U}_{\mathbf{G}}^F|\tau.$$

This completes the proof of the formula 3.5. $\square$

3.C. **On the image of $\mathrm{Cur}^{\mathbf{G}}$.** We are not able to determine in general the sub-$\mathcal{O}$-algebra $\mathrm{Im}(\mathrm{Cur}^{\mathbf{G}})$ of $\prod_{w\in W}\mathcal{O}\mathbf{T}_w^F$. Of course, we have

$$\textbf{(3.6)}\qquad \mathrm{Im}(\mathrm{Cur}^{\mathbf{G}}) \subset \mathrm{Im}({}_K\mathrm{Cur}^{\mathbf{G}}) \cap \big(\prod_{w\in W}\mathcal{O}\mathbf{T}_w^F\big).$$

However, there are cases where this inclusion is an equality:

Theorem 3.7. *If ℓ does not divide the order of W, then*

$$\mathrm{Im}(\mathrm{Cur}^{\mathbf{G}}) = \mathrm{Im}({}_K\mathrm{Cur}^{\mathbf{G}}) \cap \big(\prod_{w\in W}\mathcal{O}\mathbf{T}_w^F\big).$$

Proof. Let A be the image of $\mathrm{Cur}^{\mathbf{G}}$. Then, since $\mathcal{H}^{\mathbf{G}}$ is a symmetric algebra (with symmetrizing form $\tau_{\mathcal{H}}$), it follows from 3.5 that A is a symmetric algebra (with symmetrizing form $\tilde{\tau}_A$, the restriction of $\tilde{\tau}$ to A).

Now, let $B = \mathrm{Im}({}_K\mathrm{Cur}^{\mathbf{G}}) \cap \big(\prod_{w\in W}\mathcal{O}\mathbf{T}_w^F\big)$. If ℓ does not divide $|W|$, then the restriction of $\tilde{\tau}$ to B defines a map $\tilde{\tau}_B : B \to \mathcal{O}$. By construction, we have $A \subset B \subset KA$. So the result follows from Lemma 3.8 below. $\square$

Lemma 3.8. *Let $(\mathcal{A},\tau)$ be a symmetric $\mathcal{O}$-algebra and let $\mathcal{B}$ be a subring of $K\mathcal{A}$ such that $\mathcal{A} \subset \mathcal{B}$ and ${}_K\tau(\mathcal{B}) \subset \mathcal{O}$. Then $\mathcal{A} = \mathcal{B}$.*

Proof. Let $(a_1,\dots,a_n)$ be an $\mathcal{O}$-basis of $\mathcal{A}$ and let $(a_1^*,\dots,a_n^*)$ denote the dual $\mathcal{O}$-basis of $\mathcal{A}$ (with respect to τ). Then, for all $h \in K\mathcal{A}$, we have $h = \sum_{i=1}^n {}_K\tau(ha_i^*)a_i$. Now, if moreover $h \in \mathcal{B}$, then $ha_i^* \in \mathcal{B}$ for all i, so ${}_K\tau(ha_i^*) \in \mathcal{O}$. So $h \in \mathcal{A}$. $\square$

Remark 3.9 - If ℓ does not divide the order of W, then the Sylow ℓ-subgroups of $\mathbf{G}^F$ are abelian. If $\mathbf{G}^F = \mathbf{SL}_2(\mathbb{F}_q)$, if q is odd and if $\ell = 2$, then the inclusion 3.6 is strict. If $\mathbf{G}^F = \mathbf{GL}_3(\mathbb{F}_2)$ and if $\ell = 3$, then ℓ divides $|W|$ but the Sylow 3-subgroups of $\mathbf{G}^F$ are abelian: in this case, a brute force computation shows that the inclusion 3.6 is an equality. This suggests the following question: *do we have an equality in 3.6 if and only if the Sylow ℓ-subgroups of $\mathbf{G}^F$ are abelian?*

By Corollary 3.4 and Theorem 3.7, we get:

Corollary 3.10. *Let $t = (t_w) \in \prod_{w\in W}\mathcal{O}\mathbf{T}_w^F$ and assume that ℓ does not divide the order of W. Then $t \in \mathrm{Im}\,\mathrm{Cur}^{\mathbf{G}}$ if and only if, for all (w,θ), $(w',\theta') \in \mathcal{S}^{\mathbf{G}}$ such that $(w,\theta) \equiv (w',\theta')$, we have $\theta(t_w) = \theta'(t_{w'})$.*

Corollary 3.11. *Let $h \in K\mathcal{H}^{\mathbf{G}}$ and assume that ℓ does not divide the order of W. Then $h \in \mathcal{H}^{\mathbf{G}}$ if and only if ${}_K\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}(h) \in \mathcal{O}\mathbf{T}^F$ for all F-stable maximal tori of $\mathbf{G}$.*

The next result has been announced at the beginning of §2.C.

Corollary 3.12. *If $\mathbf{L}$ is an F-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}$ and if ℓ does not divide the order of W, then ${}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(\mathcal{H}^{\mathbf{G}}) \subset \mathcal{H}^{\mathbf{L}}$.*

Proof. Let $h \in \mathcal{H}^{\mathbf{G}}$ and let $h' = {}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(h)$. By Corollary 3.11, it is sufficient to show that ${}_K\mathrm{Cur}_{\mathbf{T}}^{\mathbf{L}}(h') \in \mathcal{O}\mathbf{T}^F$ for all F-stable maximal torus $\mathbf{T}$ of $\mathbf{L}$. But this follows from the transitivity of the Curtis maps (see Proposition 2.2) and from the fact that ${}_K\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}(\mathcal{H}^{\mathbf{G}}) \subset \mathcal{O}\mathbf{T}^F$ (see Theorem 2.7). □

3.D. **Truncation at unipotent blocks.** We keep the notation introduced in §2.D: for instance, $b^{\mathbf{G}}$ denotes the sum of the unipotent blocks of $\mathbf{G}^F$.

Theorem 3.13. *Assume that ℓ does not divide the order of W. Let S denote a Sylow ℓ-subgroup of $\mathbf{G}^F$ and let $\mathbf{T}$ denote a maximally split F-stable maximal torus of $C_{\mathbf{G}}(S)$. Then $\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}$ induces an isomorphism*

$$b^{\mathbf{G}}\mathcal{H}^{\mathbf{G}} \simeq (\mathcal{O}\mathbf{T}^F b^{\mathbf{T}})^{N_{\mathbf{G}^F}(\mathbf{T})} \simeq (\mathcal{O}S)^{N_{\mathbf{G}^F}(S)}.$$

Proof. First, since ℓ does not divide the order of W, S is contained in some maximal torus and the centralizer $C_{\mathbf{G}}(S)$ is an F-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}$. In particular, S is abelian, $\mathbf{T}^F$ contains S and S is a Sylow ℓ-subgroup of $\mathbf{T}^F$. This implies that $N_{\mathbf{G}^F}(\mathbf{T}) \subset N_{\mathbf{G}^F}(S)$. Moreover, if $n \in N_{\mathbf{G}^F}(S)$, then ${}^n\mathbf{T}$ is another maximally split maximal torus of $C_{\mathbf{G}}(S)$ so there exists $g \in C_{\mathbf{G}^F}(S)$ such that ${}^n\mathbf{T} = {}^g\mathbf{T}$. This shows that

$$N_{\mathbf{G}^F}(S) = N_{\mathbf{G}^F}(\mathbf{T}).C_{\mathbf{G}^F}(S). \tag{1}$$

This also implies that the map $\mathcal{O}S \to \mathcal{O}\mathbf{T}^F b^{\mathbf{T}}$, $x \mapsto xb^{\mathbf{T}}$ induces an isomorphism

$$(\mathcal{O}\mathbf{T}^F b^{\mathbf{T}})^{N_{\mathbf{G}^F}(\mathbf{T})} \simeq (\mathcal{O}S)^{N_{\mathbf{G}^F}(S)}.$$

So we only need to show that $\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}$ induces an isomorphism of algebras $b^{\mathbf{G}}\mathcal{H}^{\mathbf{G}} \simeq (\mathcal{O}\mathbf{T}^F b^{\mathbf{T}})^{N_{\mathbf{G}^F}(\mathbf{T})}$.

Now, by 2.8, we have that $\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}(b^{\mathbf{G}}\mathcal{H}^{\mathbf{G}}) \subset (\mathcal{O}\mathbf{T}^F b^{\mathbf{T}})^{N_{\mathbf{G}^F}(\mathbf{T})}$. So it remains to prove that $\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}$ is injective on $b^{\mathbf{G}}\mathcal{H}^{\mathbf{G}}$ and that the above inclusion is in fact an equality.

Let us first prove that $\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}$ is injective on $b^{\mathbf{G}}\mathcal{H}^{\mathbf{G}}$. Let $\mathbf{T}^*$ denote an F^*-stable maximal torus which is dual to $\mathbf{T}$. Let S^* denote the Sylow ℓ-subgroup of $\mathbf{T}^{*F^*}$. Then $|\mathbf{G}^F| = |\mathbf{G}^{*F^*}|$ and $|\mathbf{T}^F| = |\mathbf{T}^{*F^*}|$ so S^* is a Sylow ℓ-subgroup of $\mathbf{G}^{*F^*}$. In

particular, every ℓ-element of $\mathbf{G}^{*F^*}$ is conjugate to an element of S^*. So ${}_K\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}$ is injective on $b^{\mathbf{G}}K\mathcal{H}^{\mathbf{G}}$, as desired.

Moreover, since S^* is abelian, two elements of S^* are conjugate in $\mathbf{G}^{*F^*}$ if and only if they are conjugate under $N_{\mathbf{G}^{*F^*}}(S^*)$ that is, if and only if they are conjugate under $N_{\mathbf{G}^{*F^*}}(\mathbf{T}^*)$: indeed, by the same argument used above for proving (1), we have

$$(1^*) \qquad N_{\mathbf{G}^{*F^*}}(S^*) = N_{\mathbf{G}^{*F^*}}(\mathbf{T}^*).C_{\mathbf{G}^*}(S^*).$$

In particular,

$$(2) \qquad {}_K\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}(b^{\mathbf{G}}K\mathcal{H}^{\mathbf{G}}) = (K\mathbf{T}^F b^{\mathbf{T}})^{N_{\mathbf{G}^F}(\mathbf{T})}.$$

So, by (2), we only need to prove that,

(?) *if $h \in b^{\mathbf{G}}K\mathcal{H}^{\mathbf{G}}$ is such that ${}_K\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}(h) \in \mathcal{O}\mathbf{T}^F$, then $h \in b^{\mathbf{G}}\mathcal{H}^{\mathbf{G}}$.*

We shall prove (2) by induction on $\dim \mathbf{G}$, the case where $\dim \mathbf{G} = \dim \mathbf{T}$ being trivial. So let $h \in b^{\mathbf{G}}K\mathcal{H}^{\mathbf{G}}$ be such that ${}_K\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}(h) \in \mathcal{O}\mathbf{T}^F$. Let $w \in W$. By Corollary 3.11, we only need to show that ${}_K\mathrm{Cur}_{\mathbf{T}_w}^{\mathbf{G}}(h) \in \mathcal{O}\mathbf{T}_w^F$.

Let S_w denote the Sylow ℓ-subgroup of $\mathbf{T}_w^F$. Since S is a Sylow ℓ-subgroup of $\mathbf{G}^F$, we may, and we will, assume that $S_w \subset S$. Now, let $\mathbf{L} = C_{\mathbf{G}}(S_w)$. Since S_w is an ℓ-subgroup and ℓ does not divide the order of W, $\mathbf{L}$ is an F-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}$. Moreover, we have $\mathbf{T} \subset \mathbf{L}$ and $\mathbf{T}_w \subset \mathbf{L}$.

Now, let $h' = {}_K\mathrm{Cur}_{\mathbf{L}}^{\mathbf{G}}(h)$. Then $h' \in b^{\mathbf{L}}K\mathcal{H}^{\mathbf{L}}$ (see 2.8) and, by hypothesis, we have ${}_K\mathrm{Cur}_{\mathbf{T}}^{\mathbf{L}}(h') = {}_K\mathrm{Cur}_{\mathbf{T}}^{\mathbf{G}}(h) \in \mathcal{O}\mathbf{T}^F$. So, if $\dim \mathbf{L} < \dim \mathbf{G}$, then $h' \in b^{\mathbf{L}}\mathcal{H}^{\mathbf{L}}$ by induction hypothesis, and so ${}_K\mathrm{Cur}_{\mathbf{T}_w}^{\mathbf{G}}(h) = {}_K\mathrm{Cur}_{\mathbf{T}_w}^{\mathbf{L}}(h') \in \mathcal{O}\mathbf{T}_w^F$, as desired. This means that we may, and we will, assume that $\mathbf{L} = \mathbf{G}$ (or, in other words, that S_w is central in $\mathbf{G}$). This implies in particular that S_w is the Sylow ℓ-subgroup of $\mathbf{Z}(\mathbf{G})^F$. Moreover, since ℓ does not divide $|W|$, it does not divide $|\mathbf{Z}(\mathbf{G})/\mathbf{Z}(\mathbf{G})^\circ|$, so S_w is the Sylow ℓ-subgroup of $(\mathbf{Z}(\mathbf{G})^\circ)^F$. Since $|\mathbf{T}_w^F| = |\mathbf{T}_w^{*F^*}|$ and $|(\mathbf{Z}(\mathbf{G})^\circ)^F| = |(\mathbf{Z}(\mathbf{G}^*)^\circ)^{F^*}|$, the Sylow ℓ-subgroup of $\mathbf{T}_w^{*F^*}$ (which we shall denote by S_w^*) is central in $\mathbf{G}^*$.

So, let us write

$$h = \sum_{\substack{(s) \in \mathbf{G}^{*F^*}_{\mathrm{sem}}/\sim \\ s \text{ is an } \ell\text{-element}}} a_s e_{\chi_s^{\mathbf{G}}} e_\psi.$$

Then, by hypothesis,

$$\sum_{s \in S^*} a_s e_{\chi_s^{\mathbf{T}}} \in \mathcal{O}\mathbf{T}^F.$$

In other words, we have, for all $t \in \mathbf{T}^F$,

$$(3) \qquad \frac{1}{|S|} \sum_{s \in S^*} a_s \chi_s^{\mathbf{T}}(t) \in \mathcal{O}.$$

We want to show that, for all $t \in \mathbf{T}_w^F$,

$$(??) \qquad \frac{1}{|S_w|} \sum_{s \in S_w^*} a_s \chi_s^{\mathbf{T}_w}(t) \in \mathcal{O}.$$

Since $\chi_s^{\mathbf{T}_w}(t) = 1$ if t is an ℓ'-element of $\mathbf{T}_w^F$ and $s \in S_w^*$, we only need to show (??) whenever $t \in S_w$. But, in this case, $\chi_s^{\mathbf{T}_w}(t) = \chi_s^{\mathbf{T}}(t)$ since t is central in $\mathbf{G}$. On the other hand, let $S' = \{t' \in S \mid \forall\ s \in S_w^*,\ \chi_s^{\mathbf{T}}(t') = 1\}$. Then $S = S' \times S_w$. So, by (3), we have, forall $t \in S_w$,

$$\frac{1}{|S|} \sum_{t' \in S'} \Bigl(\sum_{s \in S^*} a_s \chi_s^{\mathbf{T}}(tt') \Bigr) \in \mathcal{O}.$$

But

$$\frac{1}{|S|} \sum_{t' \in S'} \Bigl(\sum_{s \in S^*} a_s \chi_s^{\mathbf{T}}(tt') \Bigr) = \frac{1}{|S_w|} \sum_{s \in S_w^*} a_s \chi_s^{\mathbf{T}_w}(t),$$

so (??) follows. $\square$

4. The Curtis-Shoji homomorphism

Let d be a fixed positive integer. In [CS, Theorem 1], Curtis and Shoji defined an algebra homomorphism from the endomorphism ring of a Gelfand-Graev representation of $K\mathbf{G}^{F^d}$ to the endomorphism ring of a Gelfand-Graev representation of $K\mathbf{G}^F$. In this section, we review the definition of this homomorphism. We conjecture that this homomorphism is defined over $\mathcal{O}$ and prove this in a special case.

Since we are working with two different isogenies F and F^d, we shall need to use more precise notation. We shall use the index $?_{(e)}$ to denote the object ? considered with respect to the isogeny F^e: for instance, $\Gamma_{(d)}^{\mathbf{G}}$ shall denote a Gelfand-Graev representation of $\mathbf{G}^{F^d}$, $\chi_{s,(1)}^{\mathbf{L}}$ shall denote the character $\chi_s^{\mathbf{L}}$ of the finite group $\mathbf{L}^F$ and so on.

4.A. **Notation.** According to our convention, the regular linear character ψ of $\mathbf{U}^F$ will be denoted by $\psi_{(1)}$. We fix a regular linear character $\psi_{(d)} : \mathbf{U}_{\mathbf{G}}^{F^d} \to \mathcal{O}^\times \subset K^\times$. Set

$$\Gamma_{(d)}^{\mathbf{G}} = \mathcal{O}\mathbf{G}^{F^d} e_{\psi_{(d)}} \simeq \operatorname{Ind}_{\mathbf{U}_{\mathbf{G}}^{F^d}}^{\mathbf{G}^{F^d}} \mathcal{O}_{\psi_{(d)}}.$$

and let $\mathcal{H}_{(d)}^{\mathbf{G}}$ denote the endomorphism algebra of the $\mathcal{O}\mathbf{G}^{F^d}$-module $\Gamma_{(d)}^{\mathbf{G}}$. For $t \in \mathbf{G}_{\mathrm{sem}}^{*F^{*d}}$, we denote by $\chi_{t,(d)}^{\mathbf{G}}$ the unique element of $\mathcal{E}(\mathbf{G}^{F^d}, (t)_{\mathbf{G}^{*F^{*d}}})$ which is an irreducible component of the character afforded by $K\Gamma_{(d)}^{\mathbf{G}}$. If $\mathbf{T}$ is an F^d-stable maximal torus, we shall denote by $\mathrm{Cur}_{\mathbf{T},(d)}^{\mathbf{G}} : \mathcal{H}_{(d)}^{\mathbf{G}} \to \mathcal{O}\mathbf{T}^{F^d}$ the Curtis homomorphism.

Remark - By Remark 1.6, the endomorphism algebra $\mathcal{H}_{(d)}^{\mathbf{G}}$ does not depend on the choice of the regular linear character $\psi_{(d)}$. There is nevertheless a "natural" choice for $\psi_{(d)}$, which is compatible with the theory of Shintani descent. It is defined as follows. Consider the map $N : \mathbf{U}_{\mathbf{G}}^{F^d}/\mathbf{D}(\mathbf{U}_{\mathbf{G}})^{F^d} \to \mathbf{U}_{\mathbf{G}}^{F}/\mathbf{D}(\mathbf{U}_{\mathbf{G}})^{F}$, $u \mapsto uF(u)\cdots F^{d-1}(u)$. Then one can take $\psi_{(d)} = \psi_{(1)} \circ N$. □

4.B. **The Curtis-Shoji homomorphism.** For an F-stable torus $\mathbf{T}$ of $\mathbf{G}$, denote by

$$\mathrm{N}_{F^d/F}^{\mathbf{T}} : \mathbf{T}^{F^d} \to \mathbf{T}^F,$$

the surjective group homomorphism

$$t \to t \cdot {}^F t \cdots {}^{F^{d-1}} t.$$

Denote by $\mathrm{N}_{F^d/F}^{\mathbf{T}}$ also the $\mathcal{O}$-linear map $\mathcal{O}\mathbf{T}^{F^d} \to \mathcal{O}\mathbf{T}^F$ extending $\mathrm{N}_{F^d/F,\mathbf{T}}$.

Proposition 4.1 (Curtis-Shoji)**.** *There exists a homomorphism of algebras*

$$_K\Delta^{\mathbf{G}} : K\mathcal{H}_{(d)}^{\mathbf{G}} \to K\mathcal{H}_{(1)}^{\mathbf{G}}$$

which is characterized as the unique linear map from $K\mathcal{H}_{(d)}^{\mathbf{G}}$ to $K\mathcal{H}_{(1)}^{\mathbf{G}}$ with the property that

$$_K\mathrm{Cur}_{\mathbf{T},(1)}^{\mathbf{G}} \circ {}_K\Delta^{\mathbf{G}} = {}_K\mathrm{N}_{F^d/F}^{\mathbf{T}} \circ {}_K\mathrm{Cur}_{\mathbf{T},(d)}^{\mathbf{G}},$$

for any F-stable torus $\mathbf{T}$ of $\mathbf{G}$.

Proof. See [CS, Theorem 1]: the proof uses essentially only the fact that the map $_K\mathrm{Cur}_{(1)}^{\mathbf{G}}$ is injective (see Proposition 3.2) and the computation of its image (see Corollary 3.4). □

Corollary 4.2. *Let $\mathbf{L}$ be an F-stable Levi subgroup of a parabolic subgroup of $\mathbf{G}$. Then $_K\Delta^{\mathbf{L}} \circ {}_K\mathrm{Cur}_{\mathbf{L},(d)}^{\mathbf{G}} = {}_K\mathrm{Cur}_{\mathbf{L},(1)}^{\mathbf{G}} \circ {}_K\Delta^{\mathbf{L}}$. In other words, the diagram*

$$\begin{tikzcd}
\mathcal{H}_{(d)}^{\mathbf{G}} \arrow[r, "{_K\mathrm{Cur}_{\mathbf{L},(d)}^{\mathbf{G}}}"] \arrow[d, "{_K\Delta^{\mathbf{G}}}"'] & \mathcal{H}_{(d)}^{\mathbf{L}} \arrow[d, "{_K\Delta^{\mathbf{L}}}"] \\
\mathcal{H}_{(1)}^{\mathbf{G}} \arrow[r, "{_K\mathrm{Cur}_{\mathbf{L},(1)}^{\mathbf{G}}}"] & \mathcal{H}_{(1)}^{\mathbf{L}}
\end{tikzcd}$$

is commutative.

Proof. Let $f = {}_K\Delta^{\mathbf{L}} \circ {}_K\mathrm{Cur}_{\mathbf{L},(d)}^{\mathbf{G}}$ and $g = {}_K\mathrm{Cur}_{\mathbf{L},(1)}^{\mathbf{G}} \circ {}_K\Delta^{\mathbf{L}}$. By Proposition 3.2, it is sufficient to show that $_K\mathrm{Cur}_{\mathbf{T},(1)}^{\mathbf{L}} \circ f = {}_K\mathrm{Cur}_{\mathbf{T},(1)}^{\mathbf{L}} \circ g$ for any F-stable maximal torus $\mathbf{T}$ of $\mathbf{L}$. But this follows from the transitivity of the Curtis homomorphisms (see Proposition 2.2) and the defining property of the homomorphisms $_K\Delta^{?}$ (see Proposition 4.1). □

We also derive a concrete formula for the map ${}_K\Delta^{\mathbf{G}}$:

Corollary 4.3. *The map ${}_K\Delta^{\mathbf{G}}$ is the unique linear map from $K\mathcal{H}_{(d)}^{\mathbf{G}}$ to $K\mathcal{H}_{(1)}^{\mathbf{G}}$ with the property that for any $t \in \mathbf{G}_{\mathrm{sem}}^{*F^{*d}}$,*

$$ {}_K\Delta^{\mathbf{G}}(e_{\chi_{t,(d)}^{\mathbf{G}}} e_{\psi_{(d)}}) = \sum_{\substack{(s)_{\mathbf{G}^{*F^*}} \in \mathbf{G}_{\mathrm{sem}}^{*F^*}/\sim_{\mathbf{G}^{*F^*}} \\ (s)_{\mathbf{G}^{*F^*}} \subset (t)_{\mathbf{G}^{*F^{*d}}}}} e_{\chi_{s,(1)}^{\mathbf{G}}} e_{\psi_{(1)}}, $$

In particular, if $(t)_{\mathbf{G}^{*F^{*d}}} \cap \mathbf{G}^{*F^*}$ is empty, then ${}_K\Delta(e_{\chi'^{\mathbf{G}}_s} e_{\psi'}) = 0$.

Proof. Let $a = {}_K\Delta^{\mathbf{G}}(e_{\chi_{t,(d)}^{\mathbf{G}}} e_{\psi_{(d)}})$ and

$$ b = \sum_{\substack{(s)_{\mathbf{G}^{*F^*}} \in \mathbf{G}_{\mathrm{sem}}^{*F^*}/\sim_{\mathbf{G}^{*F^*}} \\ (s)_{\mathbf{G}^{*F^*}} \subset (t)_{\mathbf{G}^{*F^{*d}}}}} e_{\chi_{s,(1)}^{\mathbf{G}}} e_{\psi_{(1)}}. $$

By Proposition 3.2, we only need to show that, if $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$, then ${}_K\mathrm{Cur}_{\mathbf{T},(1)}^{\mathbf{G}}(a) = {}_K\mathrm{Cur}_{\mathbf{T},(1)}^{\mathbf{G}}(b)$. But, by Proposition 4.1, we have

$$ {}_K\mathrm{Cur}_{\mathbf{T},(1)}^{\mathbf{G}}(a) = {}_K\mathrm{N}_{F^d/F}^{\mathbf{T}}({}_K\mathrm{Cur}_{\mathbf{T},(d)}^{\mathbf{G}}(e_{\chi_{t,(d)}^{\mathbf{G}}} e_{\psi_{(d)}}). $$

Therefore,

$$ {}_K\mathrm{Cur}_{\mathbf{T},(1)}^{\mathbf{G}}(a) = {}_K\mathrm{N}_{F^d/F}^{\mathbf{T}}\Big(\sum_{s \in \mathbf{T}^{F^{*d}} \cap (t)_{\mathbf{G}^{*F^{*d}}}} e_{\chi_{t,(d)}^{\mathbf{T}}} \Big). $$

On the other hand,

$$ {}_K\mathrm{Cur}_{\mathbf{T},(1)}^{\mathbf{G}}(b) = \sum_{s \in \mathbf{T}^{F^*} \cap (t)_{\mathbf{G}^{*F^{*d}}}} e_{\chi_{t,(1)}^{\mathbf{T}}}. $$

So it remains to show that, if $s \in \mathbf{T}^{F^{*d}}$, then

$$ {}_K\mathrm{N}_{F^d/F}^{\mathbf{T}}(e_{\chi_{s,(d)}^{\mathbf{T}}}) = \begin{cases} e_{\chi_{s,(1)}^{\mathbf{T}}} & \text{if } s \in \mathbf{T}^{*F^*}, \\ 0 & \text{otherwise.} \end{cases} $$

But this follows easily from the fact that, by definition [DL, 5.21.5, 5.21.6], we have $\chi_{s,(1)}^{\mathbf{T}} \circ \mathrm{N}_{F^d/F}^{\mathbf{T}} = \chi_{s,(d)}^{\mathbf{T}}$ as linear characters of $\mathbf{T}^{F^d}$. $\square$

4.C. **A map $\boldsymbol{\mathcal{H}_{(d)}^{\mathbf{G}} \to \mathcal{H}_{(1)}^{\mathbf{G}}}$.** We conjecture that, in general, ${}_K\Delta^{\mathbf{G}}(\mathcal{H}_{(d)}^{\mathbf{G}}) \subseteq \mathcal{H}_{(1)}^{\mathbf{G}}$, so that ${}_K\Delta^{\mathbf{G}}$ is defined over $\mathcal{O}$. However, we are only able to prove this in the following special case.

Theorem 4.4. *If ℓ does not divide $|W|$, then ${}_K\Delta^{\mathbf{G}}(\mathcal{H}_{(d)}^{\mathbf{G}}) \subseteq \mathcal{H}_{(1)}^{\mathbf{G}}$.*

Proof. Let $a \in \mathcal{H}^{\mathbf{G}}_{(d)}$ and let $h = {}_K\Delta^{\mathbf{G}}(a) \in K\mathcal{H}^{\mathbf{G}}_{(1)}$. By Corollary 3.11, we need to show that, if $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$, then ${}_K\mathrm{Cur}^{\mathbf{G}}_{\mathbf{T},(1)}(h) \in \mathcal{O}\mathbf{T}^F$. But, by Proposition 4.1, ${}_K\mathrm{Cur}^{\mathbf{G}}_{\mathbf{T},(1)}(h) = {}_K\mathrm{N}^{\mathbf{T}}_{F^d/F}(t)$, where $t = {}_K\mathrm{Cur}^{\mathbf{G}}_{\mathbf{T},(d)}(a)$. Now, by Theorem 3.7, ${}_K\mathrm{Cur}^{\mathbf{G}}_{\mathbf{T},(d)}(a) \in \mathcal{O}\mathbf{T}^{F^d}$. So the result follows from the fact that ${}_K\mathrm{N}^{\mathbf{T}}_{F^d/F}$ is defined over $\mathcal{O}$. ☐

4.D. **Truncating at unipotent blocks.** Here we study the restriction of the Curtis-Shoji homomorphism to the component $b^{\mathbf{G}}_{(d)}K\mathcal{H}^{\mathbf{G}}$ of $K\mathcal{H}^{\mathbf{G}}$ (by our usual convention in this section, $b^{\mathbf{G}}_{(m)}$ denotes the sum of the unipotent block idempotents of $\mathbf{G}^{F^m}$).

It is immediate from Corollary 4.3 that ${}_K\Delta(b^{\mathbf{G}}_{(d)}e_{\psi_{(d)}}) = b^{\mathbf{G}}_{(1)}e_{\psi_{(1)}}$. We denote by

$$ {}_K\Delta^{\mathbf{G}}_{\ell} : b^{\mathbf{G}}_{(d)}K\mathcal{H}^{\mathbf{G}}_{(d)} \to b^{\mathbf{G}}_{(1)}K\mathcal{H}^{\mathbf{G}}_{(1)}, $$

the map obtained by restricting ${}_K\Delta^{\mathbf{G}}$.

Proposition 4.5. *We have*

(a) ${}_K\Delta^{\mathbf{G}}_{\ell}$ *is surjective if and only if whenever a pair of ℓ-elements of $\mathbf{G}^{*F^*}$ are conjugate in $\mathbf{G}^{*F^{*d}}$, they are also conjugate in $\mathbf{G}^{*F^*}$.*

(b) ${}_K\Delta^{\mathbf{G}}_{\ell}$ *is injective if and only if every ℓ-element of $\mathbf{G}^{*F^{*d}}$ is $\mathbf{G}^{*F^{*d}}$-conjugate to an element of $\mathbf{G}^{*F^*}$.*

Proof. ${}_K\Delta^{\mathbf{G}}_{\ell}$ is a unitary map of commutative split semi-simple algebras, hence is surjective if and only if the image of any primitive idempotent is either a primitive idempotent or 0. Similarly, ${}_K\Delta^{\mathbf{G}}_{\ell}$ is injective if the image of every idempotent is non-zero. Both parts of the proposition are now immediate from Corollary 4.3. ☐

Let $\mathcal{Z}(\mathbf{G})$ denote the finite group $\mathbf{Z}(\mathbf{G})/\mathbf{Z}(\mathbf{G})^{\circ}$. The following corollary is related to [CS, Lemma 6]:

Corollary 4.6. *Let r denote the order of the automorphism of $\mathcal{Z}(\mathbf{G})_{\ell}$ induced by F. If $\gcd(d, r\ell) = 1$, then ${}_K\Delta^{\mathbf{G}}_{\ell}$ is surjective.*

Proof. The proof is somewhat similar to the proof of [CS, Lemma 6]: since our situation is a bit different and since our hypothesis is slightly weaker, we shall recall a proof. Let s and t be two ℓ-elements of $\mathbf{G}^{*F^*}$ and assume that they are conjugate in $\mathbf{G}^{*F^{*d}}$. By Proposition 4.5 (a), we only need to show that they are conjugate in $\mathbf{G}^{*F^*}$.

So let $g \in \mathbf{G}^{*F^{*d}}$ be such that $t = gsg^{-1}$. Let $A = C_{\mathbf{G}^*}(s)/C^{\circ}_{\mathbf{G}^*}(s)$ and let σ denote the automorphism of A induced by F^*. We set $\tilde{A} = A \rtimes < \sigma >$. It is a classical fact that A is an ℓ-group (since s is an ℓ-element: see for instance [BrM, Lemma 2.1]) and that there is an injective morphism $A \hookrightarrow \mathcal{Z}(\mathbf{G})^{\wedge}$ commuting with the actions of

the Frobenius endomorphisms (see for instance [B3, 8.4]). In particular, the order of σ divides r. So $\gcd(d, \tilde{A}) = 1$. Therefore, the map $\tilde{A} \to \tilde{A}$, $x \mapsto x^d$ is bijective.

Now, since s and t are F^*-stable, the element $h = g^{-1}F^*(g)$ belongs to $C_{\mathbf{G}^*}(s)$. We denote by x its class in A. The fact that g belongs to $\mathbf{G}^{*F^{*d}}$ implies that $hF^*(h)\cdots F^{*d-1}(h) = 1$. So $x\sigma(x)\cdots\sigma^{d-1}(x) = 1$. In other words, $(x\sigma)^d = \sigma^d$. So $x = 1$. In other words, $g^{-1}F^*(g) \in C^\circ_{\mathbf{G}^*}(s)$. By Lang's Theorem, this implies that s and t are conjugate in $\mathbf{G}^{*F^*}$. □

Corollary 4.7. *If ℓ does not divide $[\mathbf{G}^{F^d} : \mathbf{G}^F]$, then ${}_K\Delta_\ell^{\mathbf{G}}$ is injective.*

Proof. This follows from the fact that $|\mathbf{G}^F| = |\mathbf{G}^{*F^*}|$ (and similarly for F^d) and from proposition 4.5. □

Let us make a brief comment on this last result. If r denotes the order of the automorphism induced by F on $\mathcal{Z}(\mathbf{G})_\ell$ (as in Corollary 4.6), it is not clear if the condition that ℓ does not divide $[\mathbf{G}^{F^d} : \mathbf{G}^F]$ implies that d is prime to r. However, one can easily get the following result:

Lemma 4.8. *If ℓ divides $|\mathbf{G}^F|$ and does not divide $[\mathbf{G}^{F^d} : \mathbf{G}^F]$, then ℓ does not divide d.*

Proof. It is sufficient to show that, if ℓ divides $|\mathbf{G}^F|$, then ℓ divides $[\mathbf{G}^{F^\ell} : \mathbf{G}^F]$. For this, let $q_0 = q^{1/\delta}$ (recall that F^δ is a Frobenius endomorphism on $\mathbf{G}$ with respect to some $\mathbb{F}_q$-structure on $\mathbf{G}$). We denote by ϕ the automorphism of $V = X(\mathbf{T}) \otimes_{\mathbb{Z}} K$ such that $F = q_0\phi$. Then ϕ normalizes W so the invariant algebra $S(V^*)^W$ can be generated by homogeneous polynomials $f_1, \ldots, f_n$ (where $n = \dim_K V = \dim \mathbf{T}$) which are eigenvectors of ϕ. Let d_i denote the degree of f_i and let $\varepsilon_i \in K^\times$ be such that $\phi(f_i) = \varepsilon_i f_i$. Then

$$|\mathbf{G}^F| = q_0^{|\Phi_+|}\prod_{i=1}^n (q_0^{d_i} - \varepsilon_i)$$

and

$$|\mathbf{G}^{F^\ell}| = q_0^{\ell|\Phi_+|}\prod_{i=1}^n (q_0^{\ell d_i} - \varepsilon_i^\ell).$$

In particular, we have

$$[\mathbf{G}^{F^\ell} : \mathbf{G}^F] = q_0^{(\ell-1)|\Phi_+|}\prod_{i=1}^n (q_0^{d_i(\ell-1)} + q_0^{d_i(\ell-2)}\varepsilon_i + \cdots + q_0^{d_i}\varepsilon_i^{\ell-2} + \varepsilon_i^{\ell-1}).$$

View this last equality in $\mathcal{O}$ (and recall that $\mathfrak{l}$ denotes the maximal ideal of $\mathcal{O}$). Now, if ℓ divides $|\mathbf{G}^F|$, there exists i such that

$$q_0^{d_i} \equiv \varepsilon_i \mod \mathfrak{l}.$$

Therefore,

$$q_0^{d_i(\ell-1)} + q_0^{d_i(\ell-2)}\varepsilon_i + \cdots + q_0^{d_i}\varepsilon_i^{\ell-2} + \varepsilon_i^{\ell-1} \equiv \ell\varepsilon_i^{\ell-1} \equiv 0 \mod \mathfrak{l}.$$

This shows that $[\mathbf{G}^{F^\ell} : \mathbf{G}^F] \in \mathfrak{l} \cap \mathbb{Z} = \ell\mathbb{Z}$. $\square$

If ${}_K\Delta_\ell^{\mathbf{G}}(b_{(d)}^{\mathbf{G}}\mathcal{H}_{(d)}^{\mathbf{G}}) \subset b_{(1)}^{\mathbf{G}}\mathcal{H}_{(1)}^{\mathbf{G}}$, we denote by $\Delta_\ell^{\mathbf{G}} : b_{(d)}^{\mathbf{G}}\mathcal{H}_{(d)}^{\mathbf{G}} \to b_{(1)}^{\mathbf{G}}\mathcal{H}_{(1)}^{\mathbf{G}}$ the induced map. This happens for instance if ℓ does not divide $|W|$ (see Theorem 4.4).

Theorem 4.9. *If ℓ does not divide $|W| \cdot [\mathbf{G}^{F^d} : \mathbf{G}^F]$, then $\Delta_\ell^{\mathbf{G}} : b_{(d)}^{\mathbf{G}}\mathcal{H}_{(d)}^{\mathbf{G}} \to b_{(1)}^{\mathbf{G}}\mathcal{H}_{(1)}^{\mathbf{G}}$ is an isomorphism of algebras.*

Proof. By Theorem 4.4, the map $\Delta_\ell^{\mathbf{G}}$ is well-defined. By Corollary 4.7, it is injective. So it remains to show that it is surjective.

First, the order of $\mathcal{Z}(\mathbf{G})$ divides the order of W. So, since ℓ does not divide the order of W, we get that $\mathcal{Z}(\mathbf{G})_\ell = 1$. So, by Corollary 4.6, the map ${}_K\Delta_\ell^{\mathbf{G}}$ is surjective. So, if $h \in b_{(1)}^{\mathbf{G}}\mathcal{H}_{(1)}^{\mathbf{G}}$, there exists $\tilde{h} \in b_{(d)}^{\mathbf{G}}K\mathcal{H}_{(d)}^{\mathbf{G}}$ such that ${}_K\Delta_\ell^{\mathbf{G}}(\tilde{h}) = h$. So it remains to show that $\tilde{h} \in b_{(d)}^{\mathbf{G}}\mathcal{H}_{(d)}^{\mathbf{G}}$.

Let S be a Sylow ℓ-subgroup of $\mathbf{G}^F$. By hypothesis, it is a Sylow ℓ-subgroup of $\mathbf{G}^{F^d}$. Let $\mathbf{T}$ be a maximally split F-stable maximal torus of $C_{\mathbf{G}}(S)$ (as in Theorem 3.13). Let $\tilde{t} = {}_K\mathrm{Cur}_{\mathbf{T},(d)}^{\mathbf{G}}(\tilde{h})$ and $t = {}_K\mathrm{Cur}_{\mathbf{T},(1)}^{\mathbf{G}}(h)$. Then, by Proposition 4.1, we have

$$t = {}_K\mathrm{N}_{F^d/F}^{\mathbf{T}}(\tilde{t})$$

and, by 2.8,

$$\tilde{t} \in K\mathbf{T}^{F^d}b_{(d)}^{\mathbf{T}} \qquad \text{and} \qquad t \in K\mathbf{T}^F b_{(1)}^{\mathbf{T}}.$$

Also, by the statement (?) of the proof of Theorem 3.13, it is sufficient to show that $\tilde{t} \in \mathcal{O}\mathbf{T}^{F^d}b_{(d)}^{\mathbf{T}}$.

Write $\tilde{t} = \sum_{\tilde{z}\in\mathbf{T}^{F^d}} a_{\tilde{z}}\tilde{z}$ and $t = \sum_{z\in\mathbf{T}^F} b_z z$ with $a_{\tilde{z}} \in K$ and $b_z \in \mathcal{O}$. Let H be the kernel of the group homomorphism $\mathrm{N}_{F^d/F}^{\mathbf{T}}$. By hypothesis, S is also a Sylow ℓ-subgroup of $\mathbf{T}^{F^d}$. So ℓ does not divide $[\mathbf{T}^{F^d} : \mathbf{T}^F] = |H|$. Now, if $\tilde{z} \in \mathbf{T}^{F^d}$, and if we set $z = \mathrm{N}_{F^d/F}^{\mathbf{T}}(\tilde{z})$, then $b_z = \sum_{h\in H} a_{h\tilde{z}}$. But, since $\tilde{t} \in K\mathbf{T}^{F^d}b_{(d)}^{\mathbf{T}}$, we have $a_{h\tilde{z}} = a_{\tilde{z}}$ for every $h \in H$ (in fact, for every ℓ'-element h of $\mathbf{T}^{F^d}$). So $|H|a_{\tilde{z}} = b_z \in \mathcal{O}$, which means that $a_{\tilde{z}} \in \mathcal{O}$ since $|H|$ is invertible in $\mathcal{O}$. $\square$

Acknowledgements. The authors would like to thank Raphael Rouquier for useful discussions on the subject of this paper and especially for highlighting the importance of the existence of symmetrizing forms. They also thank Marc Cabanes and the referee for their comments and suggestions.

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